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Nonlinear strong commutativity preserving maps on skew elements of prime rings with involution

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ABSTRACT

Let $\mathcal{A}$ be a prime ring of characteristic not 2, with involution, with center $\mathcal{Z}(\mathcal{A})$ and with skew elements $\mathcal{K}$. Suppose that $f : \mathcal{K} \to \mathcal{A}$ is a map satisfying $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{K}$. Then there exists a map $\mu : \mathcal{K} \to \mathcal{Z}(\mathcal{A})$ such that $f(x) = x + \mu(x)$ for all $x \in \mathcal{K}$ or $f(x) = -x + \mu(x)$ for all $x \in \mathcal{K}$ except when $\mathcal{A}$ is an order in a 4, 9 or 16-dimensional central simple algebra.

1. Introduction

Let $\mathcal{A}$ be a ring with center $\mathcal{Z}(\mathcal{A})$. For $x, y \in \mathcal{A}$, we denote $[x, y] = xy - yx$ the commutator of x and y. We say that a map $f : \mathcal{A} \to \mathcal{A}$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for $x, y \in \mathcal{A}$. The problem of characterizing bijective additive (or linear) commutativity preserving maps is initiated by Watkins [35] in the case when $\mathcal{A}$ is a matrix algebra. Since then, the study of describing maps that preserve commutativity becomes an active research area in matrix theory, operator theory and ring theory (see for instance [1,2,5,7–9,11–13,20,22,28,30,31,33,34]).

In [4] Bell and Daif investigated a special kind of commutativity preserving maps as follows: Let $\mathcal{U}$ be a subset of $\mathcal{A}$. A map $f : \mathcal{U} \to \mathcal{A}$ is called strong commutativity preserving on $\mathcal{U}$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{U}$. Precisely, they proved that if a semiprime ring $\mathcal{A}$ admits a nonzero derivation which is

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strong commutativity preserving on a right ideal ρ of $\mathcal{A}$, then $\rho \subseteq \mathcal{Z}(\mathcal{A})$. In particular, $\mathcal{A}$ is commutative if $\rho = \mathcal{A}$. Later Brešar and Miers [6] characterized an additive map $f : \mathcal{A} \to \mathcal{A}$ which is strong commutativity preserving on the entire semiprime ring $\mathcal{A}$ and showed that f must be of the form $f(x) = \lambda x + \mu(x)$, where $\lambda \in \mathcal{C}$, $\lambda^2 = 1$ and $\mu : \mathcal{A} \to \mathcal{C}$ is an additive map where $\mathcal{C}$ is the extended centroid of $\mathcal{A}$. In [25] Ma et al. generalized Bell and Daif's result to the case of generalized derivations. On the other hand, Lin and Liu extended Brešar and Miers's result to Lie ideals and symmetric elements of prime rings in [23] and [24] respectively. Recently, more and more mathematicians are interested in discussing nonlinear commutativity preserving maps (see for instance [14–18,27,32]). In [29], Qi and Hou studied nonlinear strong commutativity preserving maps and proved that if $\mathcal{A}$ is a unital prime ring containing a nontrivial idempotent, then every nonlinear surjective strong commutativity preserving maps f on $\mathcal{A}$ must be of the form $f(x) = \lambda x + \mu(x)$ for all $x \in \mathcal{A}$, where $\lambda \in \{1, -1\}$ and $\mu : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is a map. In this paper, we continue this line of investigation and characterize nonlinear strong commutativity preserving maps f on skew elements of prime rings $\mathcal{A}$ with involution without assumptions on the existence of nontrivial idempotents in $\mathcal{A}$ or the surjectivity of f. More precisely, our main result is as follows:

Theorem 1.1. *Let $\mathcal{A}$ be a prime ring of characteristic not 2, with involution, with center $\mathcal{Z}(\mathcal{A})$ and with skew elements $\mathcal{K}$. Suppose that $f : \mathcal{K} \to \mathcal{A}$ is a map satisfying $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{K}$. Then there exists a map $\mu : \mathcal{K} \to \mathcal{Z}(\mathcal{A})$ such that $f(x) = x + \mu(x)$ for all $x \in \mathcal{K}$ or $f(x) = -x + \mu(x)$ for all $x \in \mathcal{K}$ except when $\mathcal{A}$ is an order in a* 4, 9 *or* 16*-dimensional central simple algebra.*

2. The matrix algebra case

Throughout this section, let $\mathcal{F}$ be an algebraically closed field of characteristic not 2 and let $\mathcal{A} = M_m(\mathcal{F})$ be the $m \times m$ matrix algebra over $\mathcal{F}$ with a linear involution $*$. It is known that $*$ is either the ordinary transpose or the sympletic involution [3, Corollary 4.6.13]. In this section, we determine all linear maps $f : \mathcal{K} \to \mathcal{A}$ satisfying

$$[f(x), [y, z]] + [f(y), [z, x]] + [f(z), [x, y]] = 0 \tag{2.1}$$

for all $x, y, z \in \mathcal{K}$.

We first consider $\mathcal{A} = M_m(\mathcal{F})$ under transpose involution. For $A \in M_m(\mathcal{F})$, let A^t denote the transpose of A. Then $\mathcal{K} = \{A \in M_m(\mathcal{F}) \mid A^t = -A\}$. As usual, we let $\{e_{ij} \mid 1 \leqslant i, j \leqslant m\}$ be the set of matrix units in $M_m(\mathcal{F})$. Clearly $\{e_{ij} - e_{ji} \mid 1 \leqslant i < j \leqslant m\}$ forms a basis of $\mathcal{K}$.

Lemma 2.1. *Let $\mathcal{A} = M_m(\mathcal{F})$ under transpose involution, where $m \geqslant 5$. If $f : \mathcal{K} \to \mathcal{A}$ is a linear map satisfying* (2.1)*, then there exist $\lambda \in \mathcal{F}$ and a linear map $\mu : \mathcal{K} \to \mathcal{F}$ such that $f(x) = \lambda x + \mu(x) \cdot I_m$ for all $x \in \mathcal{K}$, where I_m is the identity matrix of $\mathcal{A}$.*

Proof. Let i, j, k, ℓ, h be five distinct integers and $1 \leqslant i, j, k, \ell, h \leqslant m$. Write $f(e_{ij}-e_{ji}) = \sum_{s,t=1}^m a_{st}e_{st}$ and $f(e_{k\ell} - e_{\ell k}) = \sum_{s,t=1}^m b_{st}e_{st}$, where $a_{st}, b_{st} \in \mathcal{F}$. Setting $x = e_{ij} - e_{ji}$, $y = e_{ik} - e_{ki}$, $z = e_{k\ell} - e_{\ell k}$ in (2.1), since $[y, z] = e_{i\ell} - e_{\ell i}$, $[z, x] = 0$ and $[x, y] = e_{kj} - e_{jk}$, we obtain

$$f(e_{ij}-e_{ji})(e_{i\ell}-e_{\ell i})-(e_{i\ell}-e_{\ell i})f(e_{ij}-e_{ji})+f(e_{k\ell}-e_{\ell k})(e_{kj}-e_{jk})-(e_{kj}-e_{jk})f(e_{k\ell}-e_{\ell k}) = 0.$$

Then $e_{ii}f(e_{ij} - e_{ji})e_{i\ell} - e_{i\ell}f(e_{ij} - e_{ji})e_{\ell\ell} = 0$, $e_{hh}f(e_{ij} - e_{ji})e_{i\ell} = 0$, $e_{\ell i}f(e_{ij} - e_{ji})e_{hh} = 0$, $e_{hh}f(e_{ij} - e_{ji})e_{\ell i} = 0$ and $e_{jj}f(e_{ij} - e_{ji})e_{i\ell} + e_{jk}f(e_{k\ell} - e_{\ell k})e_{\ell\ell}$=0. Hence $a_{ii} = a_{\ell\ell}$ for all $1 \leqslant \ell \leqslant m$, $\ell \neq i, j$, $a_{hi} = a_{ih} = 0$ for all $1 \leqslant h \leqslant m$, $h \neq i, j$, $a_{h\ell} = 0$ for all $1 \leqslant h \neq \ell \leqslant m$, $h \neq i, j$, $\ell \neq i, j$ and $a_{ji} + b_{k\ell} = 0$.

Setting $x = e_{ij} - e_{ji}$, $y = e_{jk} - e_{kj}$, $z = e_{k\ell} - e_{\ell k}$ in (2.1), since $[y, z] = e_{j\ell} - e_{\ell j}$, $[z, x] = 0$ and $[x, y] = e_{ik} - e_{ki}$, we obtain

$$f(e_{ij}-e_{ji})(e_{j\ell}-e_{\ell j})-(e_{j\ell}-e_{\ell j})f(e_{ij}-e_{ji})+f(e_{k\ell}-e_{\ell k})(e_{ik}-e_{ki})-(e_{ik}-e_{ki})f(e_{k\ell}-e_{\ell k}) = 0.$$

Then $e_{jj}f(e_{ij} - e_{ji})e_{j\ell} - e_{j\ell}f(e_{ij} - e_{ji})e_{\ell\ell} = 0$, $e_{hh}f(e_{ij} - e_{ji})e_{j\ell} = 0$, $e_{\ell j}f(e_{ij} - e_{ji})e_{hh} = 0$ and $e_{ii}f(e_{ij} - e_{ji})e_{j\ell} - e_{ik}f(e_{k\ell} - e_{\ell k})e_{\ell\ell}$=0. Hence $a_{jj} = a_{\ell\ell}$ for all $1 \leqslant \ell \leqslant m$, $\ell \neq i, j$, $a_{hj} = a_{jh} = 0$ for all $1 \leqslant h \leqslant m, h \neq i, j$ and $a_{ij} - b_{k\ell} = 0$. So we obtain $f(e_{ij} - e_{ji}) = a_{ij}e_{ij} + a_{ji}e_{ji} + a_{ii} \cdot I_m$.

Recall that $a_{ji} + b_{k\ell} = 0$ and $a_{ij} - b_{k\ell} = 0$. Thus $a_{ij} = b_{k\ell} = -a_{ji}$. Consequently, $f(e_{ij} - e_{ji}) - a_{ij}(e_{ij} - e_{ji}) \in \mathcal{F} \cdot I_m$. Similarly, $f(e_{k\ell} - e_{\ell k}) - b_{k\ell}(e_{k\ell} - e_{\ell k}) \in \mathcal{F} \cdot I_m$. So we have

$$f(e_{ij} - e_{ji}) - a_{ij}(e_{ij} - e_{ji}) \in \mathcal{F} \cdot I_m \text{ and } f(e_{k\ell} - e_{\ell k}) - a_{ij}(e_{k\ell} - e_{\ell k}) \in \mathcal{F} \cdot I_m, \tag{†}$$

for all $1 \leqslant k, \ell \leqslant m, k \neq \ell$ and $\{i, j\} \cap \{k, \ell\} = \emptyset$.

Write $f(e_{12} - e_{21}) - \lambda(e_{12} - e_{21}) \in \mathcal{F} \cdot I_m$, where $\lambda \in \mathcal{F}$. Let i, j be two distinct integers and $1 \leqslant i < j \leqslant m$. Suppose first that $\{1, 2\} \cap \{i, j\} = \emptyset$. By (†), $f(e_{ij} - e_{ji}) - \lambda(e_{ij} - e_{ji}) \in \mathcal{F} \cdot I_m$. Suppose next that $\{1, 2\} \cap \{i, j\} \neq \emptyset$. Then $i = 1$ or 2. Since $m \geqslant 5$, we may choose two distinct integer k, ℓ such that $1 \leqslant k, \ell \leqslant m, k, \ell \notin \{1, 2, j\}$. Clearly, $\{1, 2\} \cap \{k, \ell\} = \emptyset$. By (†), $f(e_{k\ell} - e_{\ell k}) - \lambda(e_{k\ell} - e_{\ell k}) \in \mathcal{F} \cdot I_m$. From $\{k, \ell\} \cap \{i, j\} = \emptyset$ and (†), it follows that $f(e_{ij} - e_{ji}) - \lambda(e_{ij} - e_{ji}) \in \mathcal{F} \cdot I_m$. So we conclude that $f(x) - \lambda x \in \mathcal{F} \cdot I_m$ for all $x \in \mathcal{K}$, proving the lemma. $\square$

Next we consider $\mathcal{A} = M_m(\mathcal{F})$ under sympletic involution. In this case, it is known that $m = 2k$ and the involution $*$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$

where $A, B, C, D \in M_k(\mathcal{F})$. Thus $\mathcal{K}$ consists of all matrices of the form

$$\begin{pmatrix} A & S \\ T & -A^t \end{pmatrix}$$

where $A, S, T \in M_k(\mathcal{F})$, $S^t = S$ and $T^t = T$.

We need two simple facts.

Fact 1. *If $A, A' \in M_m(\mathcal{F})$, where $m \geqslant 2$ and $A[B, C] - [B, C]A' = 0$ for all $B, C \in M_m(\mathcal{F})$, then $A = A' \in \mathcal{F} \cdot I_m$.*

Fact 2. *If $A, A' \in M_m(\mathcal{F})$ and $A(B + B^t) - (B + B^t)A' = 0$ for all $B \in M_m(\mathcal{F})$, then $A = A' \in \mathcal{F} \cdot I_m$.*

Lemma 2.2. *Let $\mathcal{A} = M_m(\mathcal{F})$ under sympletic involution, where $m \geqslant 4$. If $f : \mathcal{K} \to \mathcal{A}$ is a linear map satisfying (2.1), then there exist $\lambda \in \mathcal{F}$ and a linear map $\mu : \mathcal{K} \to \mathcal{F}$ such that $f(x) = \lambda x + \mu(x) \cdot I_m$ for all $x \in \mathcal{K}$, where I_m is the identity matrix of $\mathcal{A}$.*

Proof. Recall that $m = 2k$ for $k \geqslant 2$. Let $\mathcal{T} = \{A \in M_k(\mathcal{F}) \mid A^t = A\}$; then $\{e_{ii}, e_{ij} + e_{ji} \mid 1 \leqslant i < j \leqslant k\}$ forms a basis of $\mathcal{T}$. For $A \in M_k(\mathcal{F})$, $S, T \in \mathcal{T}$, we write $f(\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}) = \begin{pmatrix} \varphi_1(A) & \chi_1(A) \\ \chi_2(A) & \varphi_2(A) \end{pmatrix}$, $f(\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} \phi_1(S) & \rho_1(S) \\ \rho_2(S) & \phi_2(S) \end{pmatrix}$ and $f(\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}) = \begin{pmatrix} \psi_1(T) & \eta_1(T) \\ \eta_2(T) & \psi_2(T) \end{pmatrix}$, where $\varphi_i : M_k(\mathcal{F}) \to M_k(\mathcal{F})$, $\chi_i : M_k(\mathcal{F}) \to M_k(\mathcal{F})$, $\phi_i : \mathcal{T} \to M_k(\mathcal{F})$, $\rho_i : \mathcal{T} \to M_k(\mathcal{F})$, $\psi_i : \mathcal{T} \to M_k(\mathcal{F})$, $\eta_i : \mathcal{T} \to M_k(\mathcal{F})$ are linear maps.

Set $x = \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}$, $y = \begin{pmatrix} B & 0 \\ 0 & -B^t \end{pmatrix}$ and $z = \begin{pmatrix} C & 0 \\ 0 & -C^t \end{pmatrix}$ in (2.1), where $A, B, C \in M_k(\mathcal{F})$; then we obtain

$$[\varphi_1(A), [B, C]] + [\varphi_1(B), [C, A]] + [\varphi_1(C), [A, B]] = 0 \tag{2.2}$$

and

$$[\varphi_2(A), [B^t, C^t]] + [\varphi_2(B), [C^t, A^t]] + [\varphi_2(C), [A^t, B^t]] = 0 \tag{2.3}$$

for all $A, B, C \in M_k(\mathcal{F})$. Replacing A by I_k in (2.2), we see that $[\varphi_1(I_k), [B, C]] = 0$ for all $B, C \in M_k(\mathcal{F})$. By Fact 1, $\varphi_1(I_k) \in \mathcal{F} \cdot I_k$. Similarly, using (2.3), we get $\varphi_2(I_k) \in \mathcal{F} \cdot I_k$.

Set $x = \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}$, $y = \begin{pmatrix} B & 0 \\ 0 & -B^t \end{pmatrix}$ and $z = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ in (2.1), where $A, B \in M_k(\mathcal{F})$, $S \in \mathcal{T}$; then we obtain

$$\begin{aligned} &\varphi_1(A)(BS + SB^t) - (BS + SB^t)\varphi_2(A) - \varphi_1(B)(AS + SA^t) \\ &\quad + (AS + SA^t)\varphi_2(B) + \rho_1(S)[A^t, B^t] - [A, B]\rho_1(S) = 0, \end{aligned} \tag{2.4}$$

$$-(BS + SB^t)\chi_2(A) + (AS + SA^t)\chi_2(B) + \phi_1(S)[A, B] - [A, B]\phi_1(S) = 0, \tag{2.5}$$

$$\chi_2(A)(BS + SB^t) - \chi_2(B)(AS + SA^t) + \phi_2(S)[A^t, B^t] - [A^t, B^t]\phi_2(S) = 0, \tag{2.6}$$

and

$$\rho_2(S)[A, B] = [A^t, B^t]\rho_2(S). \tag{2.7}$$

Similarly, setting $x = \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}$, $y = \begin{pmatrix} B & 0 \\ 0 & -B^t \end{pmatrix}$ and $z = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$ in (2.1), we obtain

$$(TB + B^tT)\chi_1(A) - (TA + A^tT)\chi_1(B) + \psi_2(T)[A^t, B^t] - [A^t, B^t]\psi_2(T) = 0, \tag{2.8}$$

$$-\chi_1(A)(TB + B^tT) + \chi_1(B)(TA + A^tT) + \psi_1(T)[A, B] - [A, B]\psi_1(T) = 0, \tag{2.9}$$

and

$$\eta_1(T)[A^t, B^t] = [A, B]\eta_1(T). \tag{2.10}$$

Set $x = \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}$, $y = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ and $z = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$ in (2.1), where $A \in M_k(\mathcal{F})$, $S, T \in \mathcal{T}$; then we obtain

$$\varphi_1(A)ST - ST\varphi_1(A) + \rho_1(S)(TA + A^tT) - (AS + SA^t)\eta_2(T) = 0, \tag{2.11}$$

$$-\chi_1(A)TS - ST\chi_1(A) + \psi_1(T)(AS + SA^t) - (AS + SA^t)\psi_2(T) = 0, \tag{2.12}$$

$$\chi_2(A)ST + TS\chi_2(A) + \phi_2(S)(TA + A^tT) - (TA + A^tT)\phi_1(S) = 0, \tag{2.13}$$

and

$$-\varphi_2(A)TS + TS\varphi_2(A) - (TA + A^tT)\rho_1(S) + \eta_2(T)(AS + SA^t) = 0. \tag{2.14}$$

Setting $S = T = I_k$ in (2.11), we have $\rho_1(I_k)(A + A^t) - (A + A^t)\eta_2(I_k) = 0$. By Fact 2, $\rho_1(I_k) = \eta_2(I_k) \in \mathcal{F} \cdot I_k$. Next setting $A = T = I_k$ in (2.11) and using $\varphi_1(I_k) \in \mathcal{F} \cdot I_k$, we obtain $\rho_1(S) - S\eta_2(I_k) = 0$. And setting $A = S = I_k$ in (2.11) and using $\varphi_1(I_k) \in \mathcal{F} \cdot I_k$, we obtain $\rho_1(I_k)T - \eta_2(T) = 0$. Hence $\rho_1(S) = \alpha S$ and $\eta_2(T) = \alpha T$ for all $S, T \in \mathcal{T}$, where $\alpha = \rho_1(I_k) = \eta_2(I_k) \in \mathcal{F} \cdot I_k$. With these, (2.11) reduces to $(\varphi_1(A) - \alpha A)ST - ST(\varphi_1(A) - \alpha A) = 0$. Replacing S, T by $I_k, B + B^t$ respectively, we get $(\varphi_1(A) - \alpha A)(B + B^t) - (B + B^t)(\varphi_1(A) - \alpha A) = 0$ for all $B \in M_k(\mathcal{F})$. By Fact 2, $\varphi_1(A) - \alpha A \in \mathcal{F} \cdot I_k$ for all $A \in M_k(\mathcal{F})$. Similarly, from (2.14), it follows that $\varphi_2(A) + \alpha A^t \in \mathcal{F} \cdot I_k$ for all $A \in M_k(\mathcal{F})$. Consequently, there exist linear maps $\mu_i : M_k(\mathcal{F}) \to \mathcal{F} \cdot I_k$ for $i = 1, 2$ such that $\varphi_1(A) = \alpha A + \mu_1(A)$ and $\varphi_2(A) = -\alpha A^t + \mu_2(A)$. Recall that $\rho_1(S) = \alpha S$ for all $S \in \mathcal{T}$. Now (2.4) is reduced to $(\mu_1(A) - \mu_2(A))(BS + SB^t) - (\mu_1(B) - \mu_2(B))(AS + SA^t) = 0$. Setting $A = e_{11}$ and

$B = S = e_{jj}$, where $j \neq 1$; then $(\mu_1(e_{11}) - \mu_2(e_{11}))(2e_{jj}) = 0$. Thus $\mu_1(e_{11}) - \mu_2(e_{11}) = 0$. Next setting $A = S = e_{11}$; then $-(\mu_1(B) - \mu_2(B))(2e_{11}) = 0$. This implies $\mu_1(B) = \mu_2(B) \in \mathcal{F} \cdot I_k$ for all $B \in M_k(\mathcal{F})$. Setting $A = I_k$ in (2.6); then

$$\chi_2(I_k)(BS + SB^t) - \chi_2(B)(2S) = 0. \tag{2.15}$$

Replacing B, S by e_{ji}, e_{ii} in (2.15) respectively, where $1 \leqslant i, j \leqslant k$, $i \neq j$, we get $\chi_2(I_k)(e_{ji} + e_{ij}) - \chi_2(e_{ji})2e_{ii} = 0$. Thus $\chi_2(I_k)e_{ij} = 0$ for all $1 \leqslant i, j \leqslant k$, $i \neq j$. This implies $\chi_2(I_k) = 0$. Then (2.15) becomes $\chi_2(B)(2S) = 0$ for all $B \in M_k(\mathcal{F})$ and $S \in \mathcal{T}$. Setting $S = I_k$ in (2.15); then $\chi_2 = 0$. Similarly, setting $A = I_k$ in (2.9), we obtain $\chi_1 = 0$. Consequently, we have $f(\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}) - \alpha \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \in \mathcal{F} \cdot I_m$.

From $\chi_2 = 0$, (2.5) and (2.6), it follows that $\phi_1(S)[A, B] - [A, B]\phi_1(S) = 0$ and $\phi_2(S)[A^t, B^t] - [A^t, B^t]\phi_2(S) = 0$. By Fact 1, $\phi_1(S) \in \mathcal{F} \cdot I_k$ and $\phi_2(S) \in \mathcal{F} \cdot I_k$. Now (2.13) becomes $(\phi_2(S) - \phi_1(S))(TA + A^tT) = 0$. Thus $\phi_2(S) = \phi_1(S) \in \mathcal{F} \cdot I_k$ for all $S \in \mathcal{T}$ by setting $A = T = I_k$. Similarly, from $\chi_1 = 0$, (2.8), (2.9) and (2.12), it follows that $\psi_1(T) = \psi_2(T) \in \mathcal{F} \cdot I_k$ for all $T \in \mathcal{T}$.

Fixing S in (2.7) and write $\rho_2(S) = \sum_{s,t=1}^{k} a_{st}e_{st}$. Suppose first that $k \geqslant 3$. Let i, j, ℓ be three distinct integer and $1 \leqslant i, j, \ell \leqslant k$. Replacing A, B by e_{ii}, e_{ij} in (2.7) respectively, we get $\rho_2(S)e_{ij} + e_{ji}\rho_2(S) = 0$. Hence $a_{ii} = 0$ and $a_{\ell i} = 0$. This implies $\rho_2(S) = 0$. Thus $\rho_2 = 0$. Suppose $k = 2$. Replacing A, B by e_{11}, e_{12} in (2.7) respectively, we get $\rho_2(S)e_{12} + e_{21}\rho_2(S) = 0$. Hence $a_{11} = 0$ and $a_{21} + a_{12} = 0$. Next replacing A, B by e_{11}, e_{21} in (2.7) respectively, we get $\rho_2(S)e_{21} + e_{12}\rho_2(S) = 0$, implying $a_{22} = 0$. So $\rho_2(S) = a_{12}(e_{12} - e_{21})$. Hence $\rho_2(S) \in \mathcal{F} \cdot (e_{12} - e_{21})$ for all $S \in \mathcal{T}$. Recall that $f(\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & \alpha S \\ \rho_2(S) & 0 \end{pmatrix} + \begin{pmatrix} \phi_1(S) & 0 \\ 0 & \phi_1(S) \end{pmatrix}$. Set $x = \begin{pmatrix} 0 & S_1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & S_2 \\ 0 & 0 \end{pmatrix}$ and $z = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$ in (2.1), where $S_1, S_2, T \in \mathcal{T}$; then we obtain

$$\rho_2(S_1)S_2T + TS_2\rho_2(S_1) - \rho_2(S_2)S_1T - TS_1\rho_2(S_2) = 0. \tag{2.16}$$

Write $\rho_2(e_{11}) = \gamma(e_{12} - e_{21})$, where $\gamma \in \mathcal{F}$. Replacing S_1, S_2, T by e_{11}, e_{22}, e_{22} in (2.16) respectively, we have $\rho_2(e_{11})e_{22} + e_{22}\rho_2(e_{11}) = 0$. Thus $\gamma e_{12} - \gamma e_{21} = 0$, implying $\gamma = 0$. Hence $\rho_2(e_{11}) = 0$. By symmetry, we have $\rho_2(e_{22}) = 0$. Write $\rho_2(e_{12} + e_{21}) = \mu(e_{12} - e_{21})$, where $\mu \in \mathcal{F}$. Setting $S_1 = T = e_{11}$ and $S_2 = e_{12} + e_{21}$ in (2.16), we obtain $\rho_2(e_{12} + e_{21})e_{11} + e_{11}\rho_2(e_{12} + e_{21}) = 0$, implying $-\mu e_{21} + \mu e_{12} = 0$. Thus $\mu = 0$ and then $\rho_2(e_{12} + e_{21}) = 0$. Consequently, $\rho_2 = 0$. Similarly, using (2.10), we may obtain $\eta_1 = 0$. Now we have $f(\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}) - \alpha \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \in \mathcal{F} \cdot I_m$ and $f(\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}) - \alpha \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \in \mathcal{F} \cdot I_m$. This completes the proof. $\square$

3. Main results

Throughout this section $\mathcal{A}$ is a prime ring with involution $*$, with center $\mathcal{Z}(\mathcal{A})$ and with maximal right ring of quotients $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$. The center $\mathcal{C}$ of $\mathcal{Q}$ is a field and is called the extended centroid of $\mathcal{A}$ (see [3] for details). By $\mathcal{S}$, we denote the set of symmetric elements of $\mathcal{A}$, that is, $\mathcal{S} = \{x \in \mathcal{A} \mid x^* = x\}$ and by $\mathcal{K}$, we denote the set of skew elements of $\mathcal{A}$, that is, $\mathcal{K} = \{x \in \mathcal{A} \mid x^* = -x\}$. By St_n we denote the standard identity of degree n. The involution $*$ is said to be *of the first kind* if $\alpha^* = \alpha$ for all $\alpha \in \mathcal{C}$ and *of the second kind* otherwise.

We begin with a proposition which is essential to the proof of Theorem 1.1.

Proposition 3.1. *Let $\mathcal{A}$ be a prime ring of characteristic not 2, with involution $*$, with extended centroid $\mathcal{C}$ and with skew elements $\mathcal{K}$. Suppose that $f : \mathcal{K} \to \mathcal{Q}$ is a map satisfying $[f(x), [y, z]] + [f(y), [z, x]] + [f(z), [x, y]] = 0$ for all $x, y, z \in \mathcal{K}$. Then there exist $\lambda \in \mathcal{C}$ and a map $\mu : \mathcal{K} \to \mathcal{C}$ such that*

$f(x) = \lambda x + \mu(x)$ for all $x \in \mathcal{K}$ except when $\mathcal{A}$ is an order in a 4, 9 or 16-dimensional central simple algebra.

Proof. Suppose first that $\mathcal{A}$ does not satisfy St_n for any $n \geqslant 1$. A direct expansion of $[f(x_1), [x_2, x_3]] + [f(x_2), [x_3, x_1]] + [f(x_3), [x_1, x_2]] = 0$ yields

$$f(x_1)x_2x_3 - f(x_1)x_3x_2 - x_2x_3f(x_1) + x_3x_2f(x_1) + f(x_2)x_3x_1 - f(x_2)x_1x_3$$
$$- x_3x_1f(x_2) + x_1x_3f(x_2) + f(x_3)x_1x_2 - f(x_3)x_2x_1 - x_1x_2f(x_3) + x_2x_1f(x_3) = 0.$$

Let $E, F : \mathcal{K} \times \mathcal{K} \to \mathcal{Q}$ be maps defined by $E(x, y) = f(x)y - f(y)x$ and $F(x, y) = yf(x) - xf(y)$ for $x, y \in \mathcal{K}$. Then we have

$$E(x_2, x_3)x_1 + E(x_3, x_1)x_2 + E(x_1, x_2)x_3 + x_1F(x_2, x_3) + x_2F(x_3, x_1) + x_3F(x_1, x_2) = 0$$

for all $x_1, x_2, x_3 \in \mathcal{K}$. By [10, Corollary 5.18], there exist maps $p, p', q, q' : \mathcal{K} \to \mathcal{Q}$ and $\nu : \mathcal{K} \times \mathcal{K} \to \mathcal{C}$ such that $E(x_2, x_3) = x_2p(x_3) + x_3p'(x_2) + \nu(x_2, x_3)$ and $F(x_2, x_3) = q(x_3)x_2 + q'(x_2)x_3 - \nu(x_2, x_3)$ for all $x_2, x_3 \in \mathcal{K}$. Hence $f(x_2)x_3 - f(x_3)x_2 - x_2p(x_3) - x_3p'(x_2) \in \mathcal{C}$ and $x_3f(x_2) - x_2f(x_3) - q(x_3)x_2 - q'(x_2)x_3 \in \mathcal{C}$. From [10, Corollary 5.18] it follows that there exist $a, b \in Q$ such that $f(x_2) - x_2a \in \mathcal{C}$ and $f(x_2) - bx_2 \in \mathcal{C}$. So we have $bx_2 - x_2a \in \mathcal{C}$ for all $x_2 \in \mathcal{K}$. By [10, Corollary 5.18], $a = b \in \mathcal{C}$, proving the theorem.

Suppose now that $\mathcal{A}$ satisfies St_n for some $n \geqslant 1$. By Posner's Theorem [19, Theorem 1.4.3], $\mathcal{Q} = \mathcal{AC}$ is a finite-dimensional central simple $\mathcal{C}$-algebra and $\mathcal{A}$ is an order in $\mathcal{Q}$. Denote by $\mathcal{F}$ the algebraic closure of $\mathcal{C}$. Then $\mathcal{AC} \otimes_{\mathcal{C}} \mathcal{F} \cong M_m(\mathcal{F})$ for some $m \geqslant 1$. Clearly, if $m = 1$, then $\mathcal{Q}$ is commutative. In this case, $\mathcal{Q} = \mathcal{C}$ and $f(x) \in \mathcal{C}$ for all $x \in \mathcal{K}$. Hence $f(x) = 0 \cdot x + f(x)$ for all $x \in \mathcal{K}$, proving the theorem. So now we may assume $m \geqslant 5$.

Choose a subset $\mathcal{B} = \{m_i\}_{i \in I}$ of $\mathcal{K}$ to form a basis of $\mathcal{KC}$ over $\mathcal{C}$. Define a $\mathcal{C}$-linear map $g : \mathcal{KC} \to \mathcal{AC}$ by the rule $g(\sum_i \alpha_i m_i) = \sum_i \alpha_i f(m_i)$ for $\alpha_i \in \mathcal{C}$. Since

$$\left[\sum_i \alpha_i f(m_i), \left[\sum_j \beta_j m_j, \sum_k \gamma_k m_k\right]\right] = \sum_i \sum_j \sum_k \alpha_i \beta_j \gamma_k [f(m_i), [m_j, m_k]]$$

for all $\alpha_i, \beta_j, \gamma_k \in C$, we have $[g(x), [y, z]] + [g(y), [z, x]] + [g(z), [x, y]] = 0$ for all $x, y, z \in \mathcal{KC}$. Extend g to $\mathcal{KC} \otimes_{\mathcal{C}} \mathcal{F}$ by the rule $g(\sum_i x_i \otimes \beta_i) = \sum_i g(x_i) \otimes \beta_i$ for $x_i \in \mathcal{KC}$ and $\beta_i \in \mathcal{F}$. Then g is $\mathcal{F}$-linear and $[g(x), [y, z]] + [g(y), [z, x]] + [g(z), [x, y]] = 0$ for all $x, y, z \in \mathcal{KC} \otimes_{\mathcal{C}} \mathcal{F}$.

Assume first that $*$ is of the first kind, that is, $\alpha^* = \alpha$ for all $\alpha \in \mathcal{C}$. In this case, $*$ can be extended from $\mathcal{A}$ to $\widetilde{\mathcal{A}} = \mathcal{AC} \otimes_{\mathcal{C}} \mathcal{F} \cong M_m(\mathcal{F})$ given by $(\alpha a \otimes \beta)^* = \alpha a^* \otimes \beta$ for $a \in \mathcal{A}$, $\alpha \in \mathcal{C}$ and $\beta \in \mathcal{F}$. Then $*$ is either the ordinary transpose or the sympletic involution and $\widetilde{\mathcal{K}} = \mathcal{KC} \otimes_{\mathcal{C}} \mathcal{F}$ is exactly the set of skew elements of $\widetilde{\mathcal{A}}$. Recall that $m \geqslant 5$. So $[\widetilde{\mathcal{K}}, \widetilde{\mathcal{K}}] \neq 0$. This implies $[\mathcal{K}, \mathcal{K}] \neq 0$ and hence there exist $m_1, m_2 \in \mathcal{K} \setminus \mathcal{C}$ such that m_1 and m_2 are $\mathcal{C}$-independent. Clearly, we may assume $m_1, m_2 \in \mathcal{B}$. By Lemma 2.1 and 2.2, there exists $\gamma \in \mathcal{F}$ such that $g(x) - \gamma x \in \mathcal{F}$ for all $x \in \widetilde{\mathcal{K}}$. Recall that $f(m_1) = g(m_1)$ and $f(m_2) = g(m_2)$. Thus $f(m_1) - \gamma m_1 \in \mathcal{F}$ and $f(m_2) - \gamma m_2 \in \mathcal{F}$. Moreover, $1, m_1, f(m_1) \in \mathcal{AC}$ are $\mathcal{F}$-dependent. Since $1, m_1, f(m_1) \in \mathcal{AC}$, it is easy to see that $1, m_1, f(m_1)$ are $\mathcal{C}$-dependent. Using the fact that $m_1 \notin \mathcal{C}$, we have $\gamma \in \mathcal{C}$. For any $0 \neq x \in \mathcal{K}$, either x and m_1 are $\mathcal{C}$-independent or x and m_2 are $\mathcal{C}$-independent. For short, assume x and m_1 are $\mathcal{C}$-independent. Replacing m_2 by x and proceeding with the same argument as above, we see that there exists $\gamma' \in \mathcal{C}$ such that $f(m_1) - \gamma' m_1 \in \mathcal{C}$ and $f(x) - \gamma' x \in \mathcal{C}$. From $m_1 \notin \mathcal{C}$, it follows that $\gamma = \gamma'$. Hence $f(x) - \gamma x \in \mathcal{C}$ for all $0 \neq x \in \mathcal{K}$. For $y, z \in \mathcal{K}$, we have $0 = [f(0), [y, z]] + [f(y), [z, 0]] + [f(z), [0, y]] = [f(0), [y, z]]$. Thus $[f(0), [\mathcal{K}, \mathcal{K}]] = 0$. In particular, $[f(0), [\widetilde{\mathcal{K}}, \widetilde{\mathcal{K}}]] = 0$. Since $m \geqslant 5$, we get $[f(0), \widetilde{\mathcal{K}}] = 0$. By [26, Lemma 3.1 and 3.3], $[f(0), [\widetilde{\mathcal{K}}, \widetilde{\mathcal{K}}]\widetilde{\mathcal{A}}] = 0$. From [19, Lemma 1.1.6] it follows that $f(0) \in \mathcal{F}$ and then $f(0) \in \mathcal{C}$. Consequently, $f(0) - \gamma \cdot 0 \in \mathcal{C}$, proving the theorem.

Assume now that $*$ is of the second kind. Then $\alpha^* \neq \alpha$ for some $\alpha \in \mathcal{C}$. Let $\beta = \frac{\alpha - \alpha^*}{2}$; then $0 \neq \beta \in \mathcal{C}$ and $\beta^* = -\beta$. Pick a nonzero ideal $\mathcal{I}$ of $\mathcal{A}$ such that $\beta\mathcal{I} \subseteq \mathcal{A}$ and set $\mathcal{J} = \mathcal{I}\mathcal{I}^*$. Then $\mathcal{J}$ is a $*$-ideal of $\mathcal{A}$, that is, $\mathcal{J}^* = \mathcal{J}$ and $\beta\mathcal{J} \subseteq \mathcal{A}$. Thus $\beta(\mathcal{J} \cap \mathcal{S}) \subseteq \mathcal{K}$ and then $\mathcal{J} \cap \mathcal{S} \subseteq \mathcal{K}\beta^{-1} \subseteq \mathcal{KC}$, implying that $(\mathcal{J} \cap \mathcal{S})\mathcal{C} \subseteq \mathcal{K}\mathcal{C}$. Clearly, $(\mathcal{J} \cap \mathcal{K})\mathcal{C} \subseteq \mathcal{K}\mathcal{C}$. From $2\mathcal{J} \subseteq (\mathcal{J} \cap \mathcal{S}) + (\mathcal{J} \cap \mathcal{K})$, it follows that $\mathcal{J}\mathcal{C} \subseteq \mathcal{K}\mathcal{C}$. Recall that $\mathcal{A}\mathcal{C}$ is a simple algebra. So $\mathcal{J}\mathcal{C} = \mathcal{A}\mathcal{C}$ and then $\mathcal{A}\mathcal{C} = \mathcal{K}\mathcal{C}$. Hence $[g(x), [y, z]] + [g(y), [z, x]] + [g(z), [x, y]] = 0$ for all $x, y, z \in \mathcal{K}\mathcal{C} = \mathcal{A}\mathcal{C}$. Using [22, Proposition 2.1] and [19, Lemma 1.1.8] and proceeding with the same argument as above, the proof is thereby complete. □

We are now in a position to give

Theorem 3.2. *Let $\mathcal{A}$ be a prime ring of characteristic not 2, with involution $*$, with center $\mathcal{Z}(\mathcal{A})$, with extended centroid $\mathcal{C}$ and with skew elements $\mathcal{K}$. Suppose that $f : \mathcal{K} \to \mathcal{A}$ is a map satisfying $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{K}$. Then there exists a map $\mu : \mathcal{K} \to \mathcal{Z}(\mathcal{A})$ such that $f(x) = x + \mu(x)$ for all $x \in \mathcal{K}$ or $f(x) = -x + \mu(x)$ for all $x \in \mathcal{K}$ except when $*$ is of the first kind and $\mathcal{A}\mathcal{C} \otimes_{\mathcal{C}} \mathcal{F} \cong M_m(\mathcal{F})$ under the ordinary transpose for $2 \leqslant m \leqslant 4$, where $\mathcal{F}$ denotes the algebraic closure of $\mathcal{C}$.*

Proof. If $\mathcal{A}$ is commutative, then $\mathcal{A} = \mathcal{Z}(\mathcal{A})$ and hence $f(x) - x \in \mathcal{Z}(\mathcal{A})$ for all $x \in \mathcal{K}$, proving the theorem. Thus we assume $\mathcal{A}$ is noncommutative.

Choose a subset $\mathcal{B} = \{m_i\}_{i\in I}$ of $\mathcal{K}$ to form a basis of $\mathcal{K}\mathcal{C}$ over $\mathcal{C}$. Define a $\mathcal{C}$-linear map $g : \mathcal{K}\mathcal{C} \to \mathcal{A}\mathcal{C}$ by the rule $g(\sum_i \alpha_i m_i) = \sum_i \alpha_i f(m_i)$ for $\alpha_i \in \mathcal{C}$. For $x, y \in \mathcal{K}\mathcal{C}$, write $x = \sum_i \alpha_i m_i$ and $y = \sum_j \beta_j m_j$. Then

$$\begin{aligned}
[g(x), g(y)] - [x, y] &= \left[g\left(\sum_i \alpha_i m_i\right), g\left(\sum_j \beta_j m_j\right)\right] - \left[\sum_i \alpha_i m_i, \sum_j \beta_j m_j\right] \\
&= \left[\sum_i \alpha_i f(m_i), \sum_j \beta_j f(m_j)\right] - \left[\sum_i \alpha_i m_i, \sum_j \beta_j m_j\right] \\
&= \sum_i \sum_j \alpha_i \beta_j ([f(m_i), f(m_j)] - [m_i, m_j]) = 0.
\end{aligned}$$

Extend g to $\widetilde{\mathcal{K}} = \mathcal{K}\mathcal{C} \otimes_{\mathcal{C}} \mathcal{F}$ by the rule $g(\sum_i x_i \otimes \beta_i) = \sum_i g(x_i) \otimes \beta_i$ for $x_i \in \mathcal{K}\mathcal{C}$ and $\beta_i \in \mathcal{F}$. Then g is $\mathcal{F}$-linear and

$$[g(x), g(y)] = [x, y] \text{ for all } x, y \in \widetilde{\mathcal{K}}. \tag{3.1}$$

From the Jacobi identity, it follows that $[g(x), [g(y), g(z)]] + [g(y), [g(z), g(x)]] + [g(z), [g(x), g(y)]] = 0$ for all $x, y, z \in \widetilde{\mathcal{K}}$. Since $[g(x), g(y)] = [x, y]$, we may obtain

$$[g(x), [y, z]] + [g(y), [z, x]] + [g(z), [x, y]] = 0 \text{ for all } x, y, z \in \widetilde{\mathcal{K}}. \tag{3.2}$$

We divide the proof into three cases.

Case 1. $*$ is of the second kind. In view of the proof of Proposition 3.1, $\mathcal{K}\mathcal{C} = \mathcal{A}\mathcal{C}$. Clearly, there exist $m_1, m_2 \in \mathcal{K} \setminus \mathcal{C}$ such that m_1 and m_2 are $\mathcal{C}$-independent; otherwise, $[\mathcal{K}, \mathcal{K}] = 0$ and then $0 = [\mathcal{K}\mathcal{C}, \mathcal{K}\mathcal{C}] = [\mathcal{A}\mathcal{C}, \mathcal{A}\mathcal{C}]$, implying that $\mathcal{A}$ is commutative, a contradiction. Choose a subset $\mathcal{B}$ of $\mathcal{K}$ to form a basis of $\mathcal{K}\mathcal{C}$ over $\mathcal{C}$ with $m_1, m_2 \in \mathcal{B}$. By (3.1), $[g(x), g(y)] = [x, y]$ for all $x, y \in \mathcal{K}\mathcal{C} = \mathcal{A}\mathcal{C}$. By [22, Theorem 1.1], $g(x) - \gamma x \in \mathcal{C}$ for all $x \in \mathcal{A}\mathcal{C}$, where $\gamma \in \{1, -1\}$. Recall that $f(m_1) = g(m_1)$ and $f(m_2) = g(m_2)$. Thus $f(m_1) - \gamma m_1 \in \mathcal{Z}(\mathcal{A})$ and $f(m_2) - \gamma m_2 \in \mathcal{Z}(\mathcal{A})$. For any $0 \neq x \in \mathcal{K}$, either x and m_1 are $\mathcal{C}$-independent or x and m_2 are $\mathcal{C}$-independent. For short, assume x and m_1 are $\mathcal{C}$-independent. Choose a subset $\mathcal{B}$ of $\mathcal{K}$ to form a basis of $\mathcal{K}\mathcal{C}$ over $\mathcal{C}$ with $m_1, x \in \mathcal{B}$ and proceeding with the same argument as above, we see that $f(m_1) - \gamma' m_1 \in \mathcal{Z}(\mathcal{A})$ and $f(x) - \gamma' x \in \mathcal{Z}(\mathcal{A})$, where $\gamma' \in \{1, -1\}$. From $m_1 \notin \mathcal{C}$, it follows that $\gamma = \gamma'$. Thus $f(x) - \gamma x \in \mathcal{Z}(\mathcal{A})$ for all $0 \neq x \in \mathcal{K}$. For $0 \neq y \in \mathcal{K}$, we have $0 = [0, y] = [f(0), f(y)] = [f(0), \gamma y] = \gamma[f(0), y]$. Thus $[f(0), \mathcal{K}] = 0$ and then $0 = [f(0), \mathcal{K}\mathcal{C}] = [f(0), \mathcal{A}\mathcal{C}]$. So $f(0) \in \mathcal{Z}(\mathcal{A})$. Consequently, $f(0) - \gamma \cdot 0 \in \mathcal{Z}(\mathcal{A})$, proving the theorem.

Case 2. $*$ is of the first kind and $\mathcal{A}$ is not an order in a 4, 9 or 16-dimensional central simple algebra. In this case, $*$ can be extended from $\mathcal{A}$ to $\widetilde{\mathcal{A}} = \mathcal{AC} \otimes_{\mathcal{C}} \mathcal{F}$ given by $(\alpha a \otimes \beta)^* = \alpha a^* \otimes \beta$ for $a \in \mathcal{A}$, $\alpha \in \mathcal{C}$ and $\beta \in \mathcal{F}$ and $\widetilde{\mathcal{K}} = \mathcal{KC} \otimes_{\mathcal{C}} \mathcal{F}$ is exactly the set of skew elements of $\widetilde{\mathcal{A}}$. It is known that $\widetilde{\mathcal{A}}$ is a prime ring with extended centroid $\mathcal{F}$. Clearly there exist $m_1, m_2 \in \mathcal{K} \setminus \mathcal{C}$ such that m_1 and m_2 are $\mathcal{C}$-independent; otherwise, $[\mathcal{K}, \mathcal{K}] = 0$, implying that $\mathcal{A}$ satisfies St_4 by [21, Lemma 2] and then $\mathcal{A}$ is an order in a 4-dimensional central simple algebra, a contradiction. Choose a subset $\mathcal{B}$ of $\mathcal{K}$ to form a basis of $\mathcal{KC}$ over $\mathcal{C}$ with $m_1, m_2 \in \mathcal{B}$. By (3.2), $[g(x), [y, z]] + [g(y), [z, x]] + [g(z), [x, y]] = 0$ for all $x, y, z \in \widetilde{\mathcal{K}}$. By Proposition 3.1, there is $\gamma \in \mathcal{F}$ such that $g(x) - \gamma x \in \mathcal{F}$ for all $x \in \widetilde{\mathcal{K}}$. From (3.1) it follows that $[x, y] = [g(x), g(y)] = [\gamma x, \gamma y] = \gamma^2[x, y]$. Thus $(\gamma^2 - 1)[\widetilde{\mathcal{K}}, \widetilde{\mathcal{K}}] = 0$. Recall that $[\widetilde{\mathcal{K}}, \widetilde{\mathcal{K}}] \supseteq [\mathcal{K}, \mathcal{K}] \neq 0$. Thus $\gamma^2 = 1$, implies that $\gamma = 1$ or -1. Hence $f(m_1) - \gamma m_1 \in \mathcal{Z}(\mathcal{A})$ and $f(m_2) - \gamma m_2 \in \mathcal{Z}(\mathcal{A})$, where $\gamma \in \{1, -1\}$. By the same proof as Case 1, we have $f(x) - \gamma x \in \mathcal{Z}(\mathcal{A})$ for all $0 \neq x \in \mathcal{K}$. For $0 \neq y \in \mathcal{K}$, we have $0 = [0, y] = [f(0), f(y)] = [f(0), \gamma y] = \gamma[f(0), y]$. Thus $[f(0), \mathcal{K}] = 0$. By [26, Lemma 3.1 and 3.3], $[f(0), [\mathcal{K}, \mathcal{K}]\mathcal{A}] = 0$. From [19, Lemma 1.1.6] it follows that $f(0) \in \mathcal{Z}(\mathcal{A})$. Consequently, $f(0) - \gamma \cdot 0 \in \mathcal{Z}(\mathcal{A})$, proving the theorem.

Case 3. $*$ is of the first kind and $\mathcal{A}$ is an order in a 4, 9 or 16-dimensional central simple algebra. In this case, $*$ can be extended from $\mathcal{A}$ to $\widetilde{\mathcal{A}} = \mathcal{AC} \otimes_{\mathcal{C}} \mathcal{F} \cong M_m(\mathcal{F})$, where $m = 2, 3$ or 4, $\widetilde{\mathcal{K}} = \mathcal{KC} \otimes_{\mathcal{C}} \mathcal{F}$ is exactly the set of skew elements of $\widetilde{\mathcal{A}}$ and the involution $*$ on $\widetilde{\mathcal{A}}$ is either the ordinary transpose or the sympletic involution. By assumption, it only needs to consider $\widetilde{\mathcal{A}} \cong M_m(\mathcal{F})$ under sympletic involution, where $m = 2$ or 4. If $\widetilde{\mathcal{A}} \cong M_4(\mathcal{F})$ under sympletic involution, then using Lemma 2.2, $[\widetilde{\mathcal{K}}, \widetilde{\mathcal{K}}] \neq 0$ and proceeding with the same proof as Case 2, we can easily obtain $f(x) - \gamma x \in \mathcal{Z}(\mathcal{A})$ for all $x \in \mathcal{K}$, where $\gamma \in \{1, -1\}$, proving the theorem. Now we assume $\widetilde{\mathcal{A}} \cong M_2(\mathcal{F})$ under sympletic involution. Then $\widetilde{\mathcal{K}} = \mathcal{F} \cdot (e_{11} - e_{22}) + \mathcal{F} \cdot e_{12} + \mathcal{F} \cdot e_{21}$. Clearly, $[\widetilde{\mathcal{K}}, \widetilde{\mathcal{K}}] \neq 0$. Thus $[\mathcal{K}, \mathcal{K}] \neq 0$ and there exist $m_1, m_2 \in \mathcal{K} \setminus \mathcal{C}$ such that m_1 and m_2 are $\mathcal{C}$-independent. Choose a subset $\mathcal{B}$ of $\mathcal{K}$ to form a basis of $\mathcal{KC}$ over $\mathcal{C}$ with $m_1, m_2 \in \mathcal{B}$. By (3.1), $[g(x), g(y)] = [x, y]$ for all $x, y \in \widetilde{\mathcal{K}}$. Write $g(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $g(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $g(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$, where $a_{ij}, b_{ij}, c_{ij} \in \mathcal{F}$. Using $[g(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}), g(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})] = [\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})]$, we obtain

$$a_{12}b_{21} - a_{21}b_{12} = 0, \tag{3.3}$$

$$b_{12}(a_{11} - a_{22}) - a_{12}(b_{11} - b_{22}) = 2, \tag{3.4}$$

and

$$b_{21}(a_{11} - a_{22}) - a_{21}(b_{11} - b_{22}) = 0. \tag{3.5}$$

Similarly, using $[g(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}), g(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})] = [\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})]$, we have

$$a_{12}c_{21} - a_{21}c_{12} = 0, \tag{3.6}$$

$$a_{12}(c_{11} - c_{22}) - c_{12}(a_{11} - a_{22}) = 0, \tag{3.7}$$

and

$$a_{21}(c_{11} - c_{22}) - c_{21}(a_{11} - a_{22}) = -2. \tag{3.8}$$

From $[g(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}), g(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})] = [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})]$ it follows that

$$b_{12}c_{21} - b_{21}c_{12} = 1, \tag{3.9}$$

$$b_{12}(c_{11} - c_{22}) - c_{12}(b_{11} - b_{22}) = 0 \tag{3.10}$$

and

$$b_{21}(c_{11}-c_{22})-c_{21}(b_{11}-b_{22})=0. \tag{3.11}$$

Suppose that $a_{12} \neq 0$. By (3.3), $b_{21} = \frac{a_{21}b_{12}}{a_{12}}$. With this, (3.5) becomes $a_{21}(b_{12}(a_{11}-a_{22}) - a_{12}(b_{11}-b_{22})) = 0$. Then by (3.4), we have $2a_{21} = 0$, implying that $a_{21} = 0$. So $a_{12}b_{21} = 0$ by (3.3) and $a_{12}c_{21} = 0$ by (3.6). Thus $b_{21} = 0$ and $c_{21} = 0$. But $b_{12}c_{21} - b_{21}c_{12} = 1$ by (3.9), a contradiction. Hence $a_{12} = 0$. So by (3.3) and (3.4),

$$a_{21}b_{12}=0 \text{ and } b_{12}(a_{11}-a_{22})=2.$$

If $a_{21} \neq 0$, then $b_{12} = 0$, contradicting the fact that $b_{12}(a_{11}-a_{22}) = 2 \neq 0$. Thus $a_{21} = 0$. Then by (3.5), (3.7) and (3.8), we have

$$b_{21}(a_{11}-a_{22})=0, \;\; c_{12}(a_{11}-a_{22})=0 \text{ and } c_{21}(a_{11}-a_{22})=2.$$

From $b_{12}(a_{11}-a_{22}) = c_{21}(a_{11}-a_{22}) = 2$ it follows that $a_{11}-a_{22} \neq 0$ and $b_{12} = c_{21}$. Using $b_{21}(a_{11}-a_{22}) = c_{12}(a_{11}-a_{22}) = 0$, we obtain $b_{21} = c_{12} = 0$. Then by (3.9), (3.10) and (3.11),

$$b_{12}c_{21}=1, \;\; b_{12}(c_{11}-c_{22})=0 \text{ and } c_{21}(b_{11}-b_{22})=0.$$

Using $b_{12} = c_{21}$, $b_{12}(a_{11}-a_{22}) = 2$ and $b_{12}c_{21} = 1$, it is easy to see that $b_{12}^2 = 1$ and $a_{11}-a_{22} = 2b_{12}$. Finally, from $b_{12}(c_{11}-c_{22}) = c_{21}(b_{11}-b_{22}) = 0$ it follows that $c_{11} = c_{22}$ and $b_{11} = b_{22}$. So we have $g(x) - \gamma x \in \mathcal{F} \cdot I_2$ for all $x \in \widetilde{\mathcal{K}}$, where $\gamma = b_{12} \in \{1, -1\}$. In particular, $f(m_1) - \gamma m_1 \in \mathcal{Z}(\mathcal{A})$ and $f(m_2) - \gamma m_2 \in \mathcal{Z}(\mathcal{A})$. By the same proof of Case 1, we have $f(x) - \gamma x \in \mathcal{Z}(\mathcal{A})$ for all $0 \neq x \in \mathcal{K}$. For $0 \neq y \in \mathcal{K}$, we have $0 = [0, y] = [f(0), f(y)] = [f(0), \gamma y] = \gamma[f(0), y]$. Thus $[f(0), \mathcal{K}] = 0$. In particular, $[f(0), \widetilde{\mathcal{K}}] = 0$. Write $f(0) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$, where $d_{ij} \in \mathcal{F}$. Using $[\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}] = 0$, we obtain $d_{11} = d_{22}$ and $d_{21} = 0$. Using $[\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}] = 0$, we get $d_{12} = 0$. Thus $f(0) \in \mathcal{F}$, implying $f(0) \in \mathcal{Z}(\mathcal{A})$. Consequently, $f(0) - \gamma \cdot 0 \in \mathcal{Z}(\mathcal{A})$, proving the theorem. $\square$

We illustrate the exceptional cases in Theorem 3.2 as follows.

Example. Let $\mathcal{A} = M_4(\mathcal{F})$ under the ordinary transpose involution, where $\mathcal{F}$ is a field of characteristic not 2. Then $\mathcal{K}$ is $\mathcal{F}$-linear spanned by $\{e_{ij} - e_{ji} \mid 1 \leqslant i < j \leqslant 4\}$. Suppose that $f : \mathcal{K} \to \mathcal{A}$ is a $\mathcal{F}$-linear map defined by $f(e_{12}-e_{21}) = e_{34}-e_{43}$, $f(e_{13}-e_{31}) = -(e_{24}-e_{42})$, $f(e_{14}-e_{41}) = e_{23}-e_{32}$, $f(e_{23}-e_{32}) = e_{14}-e_{41}$, $f(e_{24}-e_{42}) = -(e_{13}-e_{31})$ and $f(e_{34}-e_{43}) = e_{12}-e_{21}$. Then f is strong commutativity preserving on $\mathcal{K}$ but f can not be expressed as the form given in Theorem 1.1.

Example. Let $\mathcal{A} = M_3(\mathcal{F})$ under the ordinary transpose involution, where $\mathcal{F}$ is a field of characteristic not 2. Then $\mathcal{K}$ is $\mathcal{F}$-linear spanned by $\{e_{ij} - e_{ji} \mid 1 \leqslant i < j \leqslant 3\}$. Suppose that $f : \mathcal{K} \to \mathcal{A}$ is a $\mathcal{F}$-linear map defined by $f(e_{12}-e_{21}) = e_{12}+e_{21}$, $f(e_{13}-e_{31}) = -(e_{13}+e_{31})$, and $f(e_{23}-e_{32}) = e_{23}+e_{32}$. Then f is strong commutativity preserving on $\mathcal{K}$ but f can not be expressed as the form given in Theorem 1.1.

Example. Let $\mathcal{A} = M_2(\mathcal{F})$ under the ordinary transpose involution, where $\mathcal{F}$ is a field of characteristic not 2. Then $\mathcal{K} = \mathcal{F} \cdot (e_{12}-e_{21})$. Suppose that $f : \mathcal{K} \to \mathcal{A}$ is a $\mathcal{F}$-linear map defined by $f(e_{12}-e_{21}) = e_{12}$. Then f is strong commutativity preserving on $\mathcal{K}$ but f can not be expressed as the form given in Theorem 1.1.

Finally, using Brešar and Miers's result [6] and modifying the proof of Theorem 3.2 in the case when $*$ is of the second kind, we can easily obtain

Theorem 3.3. *Let $\mathcal{A}$ be a prime ring with center $\mathcal{Z}(\mathcal{A})$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a map satisfying $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{A}$. Then there is a map $\mu : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ such that $f(x) = x + \mu(x)$ for all $x \in \mathcal{A}$ or $f(x) = -x + \mu(x)$ for all $x \in \mathcal{A}$.*

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